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Propositional Systems in Local Field Theories

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We investigate propositional systems for local field theories, which reflect intrinsically the uncertainties of measurements made on the physical system, and satisfy the isotony and local commutativity postulates of Haag and Kastler. The space-time covariance can be implemented in a natural way in these propositional systems. New techniques are introduced to obtain these propositional systems: the lattice-valued logics. The decomposition of the complete orthomodular lattice-valued logics shows that these logics are more general than the usual two-valued ones and that in these logics there is enough structure to characterize the classical and quantum, nonrelativistic and relativistic local field theories in a natural way. The Hilbert modules give the natural inner product "spaces" (modules) for the realization of the lattice-valued logics.

1. INTRODUCTION

It is a well-known fact that the relativistic quantum field theory used nowadays possesses many mathematically as well as physically unsatisfactory features. To solve these difficulties, at least partly, we think that one must find, first of all, a mathematically well-defined kinematical picture in which we can then implement in order the dynamical principles. Such a kinematical picture has to be in a simple and clear connection with the measuring processes made on physical fields (or on observables).

In such measuring processes are uncertainties of principle which can be reduced ideally to two components:

(1) The measurings of two *different observables* on a (small) region of the physical space Ω (or, ideally, at a point $x \in \Omega$) can disturb each other. (In nonrelativistic quantum mechanics this is the only case considered.)

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(2) The measurings of observables in two *different regions* of the physical space Ω (or, ideally, at two different points $x_1, x_2 \in \Omega$) can disturb each other. (This is characteristic for a relativistic quantum field theory.)

Einstein causality (or local commutativity), which refers to the second case, restricts the possible regions (or points) to the lightlike or timelike separated ones [in the nonrelativistic case the measurements do not interfere with each other in disjoint regions $S_1, S_2 \subset \Omega \subseteq \mathbb{R}^3$, $S_1 \cap S_2 = \varnothing$, thus essentially the uncertainty under (1) appears only in this case].

These two types of noncompatibilities must be reflected intrinsically in the mathematical structure of the wanted kinematical picture of local fields. To find a good kinematical picture, our starting point was the quantum logic approach to axiomatic quantum theory (Birkhoff and von Neumann, 1936; Mackey, 1963; Jauch, 1968; Gudder, 1970; Piron, 1976).

At present there is not yet known a generalization of this approach to field theory. It seems that the existing methods and techniques are inadequate to fill up this gap, so we introduce a new technique, namely, the lattice-valued logics, which, we think, does the trick. Briefly, after a short consideration of classical local field theory we introduce Boolean-valued propositions in the classical cases. Then we generalize this Boolean-valued logic to a complete orthomodular lattice-valued logic. The study of lattice-valued logics shows that there are in these logics enough structure to characterize in a simple and clear way the different (classical and quantum, nonrelativistic and relativistic) field theoretic cases. We show briefly the connection between this approach and the C^*-algebraic approach of Haag and Kastler (1964). Further we give examples for the realizations of lattice-valued logics and lattice-valued propositional systems by Hilbert modules. (A Hilbert module is a module over a *-algebra and it has similar properties as a Hilbert space (see for definition Banai, 1978)).

We hope that these new techniques and methods will be used profitably and usefully in the study of C^* and W^* algebras and Segal algebras as well as in quantum set theory (of Takeuti, 1979) and lead us nearer to a well-defined analytical formalism for the numerical calculation of relativistic quantum field theory (see Banai, 1980a, b).

We note that the work of Banai (1978) served as background of the ideas of this work, and the main concepts and results of this paper have been drawn from the doctoral thesis of the author (Banai, 1980c).

2. PHYSICAL ASSUMPTIONS AND PROPOSITIONS IN CLASSICAL LOCAL FIELD THEORIES

We consider a physical system $P(\Omega)$ spread over a physical space region Ω ($\Omega \subseteq \mathbb{R}^3$ in the nonrelativistic case, or $\Omega \subseteq \mathbb{M}^4$ in the relativistic case).

Depending on the choice of the subset Ω, the physical system $P(\Omega)$ is described by the tools of the

(a) point mechanics (Ω: $=n$ points in $\mathbb{R}^3$, respectively, word lines in $\mathbb{M}^4$),
(b) statistical mechanics ($n \gg 1$), or
(c) local field theory (Ω is a domain, but it can be the whole space, too).

In what follows we shall consider mainly the cases under (c). Let $A(\Omega)$ denote a measuring apparatus which covers Ω or a subset of Ω and can interact with $P(\Omega)$ on Ω or a subset of Ω. This can be implemented, for example, so that one places densely (in the ideal case at all points) measuring apparatuses (and observers) on Ω to be able to measure all parts of $P(\Omega)$. Then the complete collection of these apparatuses means the measuring apparatus for us.

The measure of a physical observable F means a specific interaction characteristic for F, between $P(\Omega)$ and $A(\Omega)$. This interaction modifies the interacting "surface" of $A(\Omega)$ and this deformation, which is a function on Ω, gives the measured value of F. One can determine this deformation by the above-mentioned way that the observers measure the change at all points of Ω.

We can distinguish essentially two types of observables of a $P(\Omega)$ from the measuring point of view:

(1) Global observables, which characterize the whole $P(\Omega)$ and whose values are independent of the points of Ω (e.g., total energy).

(2) Local observables which characterize $P(\Omega)$ (point by point) at the points of Ω and whose values depend on $x \in \Omega$ (e.g., energy density).

A wide class of global observables can perform by means of local ones. More exactly: A global observable F can be defined to all local observables $\mathcal{F}(x)$, which have integrable functions $f(x)$ as measured values, with the integral

$$f = \int_\Omega f(x)\, d\mu(x)$$

where f denotes the measured value of F and $\mu(x)$ is a measure characteristic for F. Written in symbols

$$F = \int_\Omega \mathcal{F}(x)\, d\mu(x)$$

For example: In the classical field theory, the energy-momentum four-vector P_μ is determined by the integral

$$P_\mu = \int_\sigma T_{\mu\nu}(x)\, d\sigma^\nu(x)$$

where $T_{\mu\nu}(x)$ is the energy-momentum tensor field.

Thus we will consider mainly local observables of a $P(\Omega)$, in what follows.

The space of all the possible values of the observables [the observation space of Birkhoff and von Neumann (1936)] is a subspace of $\mathbb{R}^N$ for global observables and a submodule of $R^N(\Omega)$ for local observables, where $R(\Omega)$ is a real-valued function space on Ω.

The experimental propositions of a physical system $P(\Omega)$ correspond to the subsets (of certain type) of the observation space. The Borel subsets of $\mathbb{R}^N$, for example, are such subsets in the case of global observables. To avoid technical difficulties, we shall consider, as experimental propositions, the subsets $S^N(\Omega)$ of $R^N(\Omega)$ of the form

$$S^N(\Omega):=[a,b)^N(\Omega):=\{f_i\in R_i(\Omega), a_i\leqslant f_i(x)<b_i,$$
$$\times\,\forall x\in\Omega,\ a_i, b_i\in\mathbb{R},\ i=1,\ldots,N\}$$

If we choose a proposition (i.e., a subset from the observation space), then the measured values of a sequence of global observables either coincide under the corresponding measurement with an element of the chosen subset or not. But in the case of local observables, if $f(x)=(f_1(x),\ldots,f_N(x))$ denotes the measured value of a sequence $F_1(x),\ldots,F_N(x)$ of local observables and $S^N(\Omega)$ is the proposition considered, then we can find under the measurement the following three possibilities:

(a) $f(x)$ is completely outside $S^N(\Omega):=[a,b)^N(\Omega)$,
(b) $f(x)$ is inside $S^N(\Omega)$ for a subset ω of Ω and is outside $S^N(\Omega)$ for $C\omega$, ($C\omega:=\Omega-\omega$ is the set-theoretic complement of ω),
(c) $f(x)$ is completely inside $S^N(\Omega)$.

In the usual sense, the proposition $S^N(\Omega)$ is true in case (c) and false in cases (a) and (b). Thus we lose the information content of case (b) provided by the measurement, if we only allow the true and false value for the propositions. This would mean that these propositions are *not* the simplest corresponding to the measurements. To preserve *all* information of the measurement in the simplest way, we give a new logical value for the case (b) and call it the *true–false value*. Also one could say: a proposition $S^N(\Omega)$ is

(x) false if its value is the empty set $\varnothing$,
(xx) true–false if its value is a subset ω of Ω; $S^N(\Omega)$ is true on ω and false on $C\omega$,
(xxx) true if its value is the whole set Ω.

This means that the propositions of a classical physical system $P(\Omega)$, in general, take their values in the power set $\mathcal{P}(\Omega)$ of the set Ω. If one regards only the measurable propositions of a $P(\Omega)$, then they have values in the Borel sets $B(\Omega)$ of Ω. Thus we are led to consider the system of propositions of a classical physical system $P(\Omega)$ as a *Boolean-valued logic* (Jech, 1971; Takeuti, 1973).

3. THE SYSTEMS OF PROPOSITIONS AS LATTICE-VALUED LOGICS

We follow the conclusion of the preceding section and we suppose that, in the most general cases [e.g., in the relativistic quantum cases, keeping in mind the two components (1) and (2) of the uncertainties of the measurements mentioned in the Introduction], the set $l=\{a, b, c, \ldots\}$ of values of the local propositions of a general physical system $P(\Omega)$ is a *complete orthomodular lattice* with union $\bigvee$, intersection $\bigwedge$ and orthocomplementation $'$.[2]

Furthermore we shall call a *local proposition* (briefly, proposition) every experiment leading to an alternative of which the terms are the elements of a complete orthomodular lattice l with maximal element 1 and minimal element 0.

Let $L=\{A, B, C, \ldots\}$ be the system of (local) propositions of a general $P(\Omega)$. One can easily impose axioms on L, close, as much as possible, to those of a usual quantum mechanical logic. We follow here closely the works of Gudder (1970) and Piron (1976).

Let us take an appropriate set of such axioms.

Whenever a proposition A takes a value a in l it follows that a proposition B takes a value b in l such that $a \leqslant b$ and we say A implies B, or in symbols, $A \subseteq B$. This relation should satisfy:

(A1) $\qquad A \subseteq A, \quad \forall A \in L$

(A2) $\qquad A \subseteq B, \quad B \subseteq C \Rightarrow A \subseteq C$

(A3)[3] $\qquad A \subseteq B, \quad B \subseteq A \Rightarrow A = B$

Thus $(L, \subseteq)$ is a partially ordered set and the least upper bound (LUB)

[2] Behind this choice (i.e., that l is not Boolean in general) is the clear (intuitive) expectation that we shall be able to describe in the value lattice of the propositions the second type of the uncertainties of measurements (because, as we saw in the previous section, the values of the propositions are in direct connection with the subsets of Ω) while we will describe the first type of the uncertainties in the system of (local) propositions.

[3] We note that (A3) shows that a proposition means for us an equivalence class of questions of Piron (1976).

and the greatest lower bound (GLB) can be defined in the usual way (see for example Gudder, 1970). We denote them by $\cup$ and $\cap$, respectively. The LUB and GLB exist *not* necessarily in $(L, \subseteq)$.[4] For mathematical convenience and simplicity we shall require now

(A4) For any family $A_i \in L$, the GLB $\bigcap_i A_i$ exists in L and its value is $\bigwedge_i a_i$ in l, where a_i is the value of A_i.

Now the logical negation corresponds to orthocomplementation: for $A \in L$ we assume that there is an $\bar{A} \in L$ which is true whenever A is false and true–false whenever A is false–true; if a is the value of A then a' is the value of $\bar{A}$. We call $\bar{A}$ the orthocomplement of A and we postulate that the map $A \to \bar{A}$ satisfies

(A5) $\overline{(\bar{A})} = A, \qquad \forall A \in L$

(A6) $A \subseteq B \Rightarrow \bar{B} \subseteq \bar{A}$

(A7) $A \cap \bar{A} = \theta, \qquad \forall A \in L$ (θ is the minimal element of L)[5]

It immediately follows that L is a complete orthocomplemented lattice with the unique minimal element θ and maximal element $\mathbb{1}$. The only possible values of θ and $\mathbb{1}$ are the 0 and 1 elements of l, respectively.

We postulate further that

(A8) L is weakly modular:

$$A \subseteq B \Rightarrow (B \cap \bar{A}) \cup A = B, \qquad \forall A, B \in L$$

Thus L is a complete orthomodular lattice, i.e., a CROC in the terminology of Piron (1976).

We say that $A \in L$ is orthogonal (or disjoint) to B if $A \subseteq \bar{B}$, in write $A \perp B$. We can now define the notion of compatibility either with Gudder's (1970) (and others') or with Piron's (1976) (and others') definition.

[4] Piron (1976) proves that the set of propositions is a complete lattice (Theorem 2.1), but he supposes under the proof that the set of questions is a complete lattice. Thus he only proves that, if the set of questions is a complete lattice then the equivalence classes of questions (propositions) are a complete lattice.

[5] We note that (A7) shows that we can redefine every proposition A as a two-valued one; A is true if its value is a and false if its value is a'. (But it is clear that the set of values of A cannot be identified with the set $\{0, 1\}$.) The author is indebted to D. Finkelstein for calling his attention to this fact.

Definition 1. Two propositions A and B in L are compatible if there exist mutually orthogonal propositions A_1, B_1, C such that $A = A_1 \cup C$, $B = B_1 \cup C$.

Definition 1'. Two propositions A and B, in L are said to be compatible if the sublattice generated by $\{A, \bar{A}, B, \bar{B})$ is distributive.

We denote this property by $A \leftrightarrow B$.

These two definitions of compatibility are equivalent for a weakly modular orthocomplemented lattice [as can easily be verified; see Banai (1980c)]. This shows the prominent practical role of the weak modularity axiom, too.

We note that the basic rules of the propositional calculus (Piron, 1976) and those theorems which follow directly from these rules, remain valid because L is a CROC.

The compatibility is the exact formulation of the simultaneous measurability in the system of propositions (Gudder, 1970). We assumed in the axioms of partial ordering that A and B are simultaneously testable if $A \subseteq B$. Also to avoid contradiction, the following statement must hold in L.

Lemma 2. If $A \subseteq B$ then $A \leftrightarrow B$.

Proof. It follows from (A8) that $B = (B \cap \bar{A}) \cup A$, but $(B \cap \bar{A}) \cup A$ and θ is orthogonal to every other proposition. Hence $B = (B \cap \bar{A}) \cup A$, $A = \theta \cup A$. Then $A \leftrightarrow B$ according to Definition 1. ∎

Further, any $A \in L$ must be compatible with $\bar{A} \in L$, that is satisfied, in fact.

The partial ordering and orthocomplementation were defined by the values of the propositions and by the corresponding structure of l. So it is useful to define a value function which assigns to an element of L its value and which is a map $L \to l$ and preserves the structure of L. Further, it maps the maximal element of L onto the maximal element of l. We must expect physically that a value function maps compatible elements in L onto compatible elements in l. Thus a value function is, in particular, a unitary c-morphism of Piron (1976) from L to l. Therefore we have the following definition.

Definition 3. A mapping v from a CROC L onto a CROC l is called a value function if

(a) $$v\left(\bigcap_i A_i\right) = \bigwedge_i v(A_i)$$

(b) $$A \subseteq \bar{B} \Leftrightarrow v(A) \leqslant v(B)'$$

(c) $$v(\mathbb{1}) = 1$$

Thus we are able to define a lattice-valued logic in an appropriate way.

Definition 4. We call the triple (L, l, V) a *CROC-valued logic* if L and l are CROCs and V is the nonempty class of value functions from L onto l such that $\forall a \in l \exists A^a \in L\ v(A^a)=a$, $\forall v \in V$. It is clear that the element A^a is unique.

Let us see now the compatibility relation between L and l.

Lemma 5. (a) $A, B \in L$, $A \leftrightarrow B \Rightarrow v(A) \leftrightarrow v(B)$; (b) $v(A) \perp v(B)$ in $l \Rightarrow A \leftrightarrow B$ in L.

Proof. (a) By Definition 1 $A \leftrightarrow B \Rightarrow \exists A_1, B_1, C \in L$ mutually orthogonal such that $A=A_1 \cup C$, $B=B_1 \cup C$. Then $v(A)=v(A_1) \vee v(C)$, $v(B)=v(B_1) \vee v(C)$. Since $A_1 \perp C$, $B_1 \perp C$, $A_1 \perp B_1 \Rightarrow v(A_1) \perp v(C)$, $v(B_1) \perp v(C)$, $v(A_1) \perp v(B_1)$.

(b) $v(A) \perp v(B) \Leftrightarrow v(A) \leqslant v(B)' \Leftrightarrow A \subseteq \bar{B} \Leftrightarrow A \perp B \Rightarrow A \leftrightarrow B$. ■

Corollary 6. If $\mathfrak{a}, \mathfrak{b} \subset l$, $\mathfrak{a} \perp \mathfrak{b}$ and $\mathfrak{a}=v(\mathcal{A})$, $\mathfrak{b}=v(\mathcal{B})$, $\mathcal{A}, \mathcal{B} \subset L$ then $\mathcal{A} \leftrightarrow \mathcal{B}$, $v \in V$. The image of a Boolean sublattice of L under a $v \in V$ is a Boolean sublattice in l and the image of a maximal Boolean sublattice of L is a maximal Boolean sublattice of l.

We call the pair $(\mathcal{C}, \mathfrak{c})$ the *center pair* of a CROC-valued logic (L, l, V) if $\mathcal{C}$ and $\mathfrak{c}$ are the centers of L and l, respectively. Then we have the following lemma.

Lemma 7. In a CROC-valued logic (L, l, V), $v(\mathcal{C})=\mathfrak{c}$, $v \in V$ and $(\mathcal{C}, \mathfrak{c}, v|_{\mathcal{C}})$ is a Boolean CROC-valued logic (i.e., $\mathcal{C}$ and $\mathfrak{c}$ are Boolean).

Proof. A $v \in V$ preserves the compatibility and is surjective; thus we have $A \in \mathcal{C} \Leftrightarrow A \leftrightarrow \forall B \in L \Rightarrow v(A) \leftrightarrow \forall v(B) \in l \Rightarrow v(A) \in \mathfrak{c} \Rightarrow v(\mathcal{C}) \subseteq \mathfrak{c}$. Now if $a \in \mathfrak{c}$ is arbitrary we have to prove that $\exists A \in \mathcal{C}$ such that $v(A)=a$. It is clear that the $A=A^a$ is such an element in $\mathcal{C}$. For, if $a \in \mathfrak{c} \Rightarrow a' \in \mathfrak{c}$, consider the following: (1) $B \in L$ and $v(B) \leqslant a' \Rightarrow v(B) \perp a \Rightarrow B \leftrightarrow A^a$; (2) $B \in L$ and $v(B) \leqslant a \Rightarrow B \subseteq A^a \Rightarrow B \leftrightarrow A^a$; (3) $B \in L$ and $v(B)=b$ is arbitrary. Since $a \in \mathfrak{c}$, $a \leftrightarrow b \Rightarrow a=a_1 \vee c$, $b=b_1 \vee c$, a_1, b_1, c are mutually orthogonal; further, $c=a \wedge b \leqslant a$ and $b_1=b \wedge (a' \vee b')=a' \wedge b \leqslant a'$ (see Gudder, 1970, Lemma 4.2; Piron, 1976, Theorem 2.19) $\Rightarrow B \subseteq A^{b_1} \cup A^c$, $A^{b_1} \perp A^c \Rightarrow B \leftrightarrow A^a=A^{a_1} \cup A^c$.

Now $\mathcal{C}$ and $\mathfrak{c}$ are Boolean CROCs (Piron, 1976) and $v(\mathcal{C})=\mathfrak{c}$, $\forall v \in V$, thus $(\mathcal{C}, \mathfrak{c}, V|_{\mathcal{C}})$ is a Boolean CROC-valued logic. ■

Finally we impose the atomicity axion on L.

If $A \neq B$, A, $B \in L$, and $A \subseteq B$, one says that B covers A when $A \subseteq X \subseteq B \Rightarrow X=A$ or $X=B$. An element which covers θ is called an *atom*.

We require the following:

(A9) If $A \in L$, $A \neq \theta$, then $\exists P \in L$ an atom such that $P \subseteq A$, (i.e., L is an atomic lattice).

(A10) If $P \in L$ is an atom, $B \in L$, $P \cap B = \theta$, then $P \cup B$ covers B.

Thus a set of lattice-valued propositions L satisfying axioms (A1)–(A10) is a propositional system in the sense of Piron (1976, p. 25).

Definition 8. A CROC-valued logic (L, l, V) is called a *CROC-valued propositional system* whenever L and l are propositional systems. A CROC-valued propositional system is called classical if L and l are distributive (Boolean).

Remark. It is well known that Piron determined the realization of the axioms (A1)–(A10), in the usual two-valued cases (except some special cases), with (generalized) Hilbert spaces (Piron, 1976). This is the practical reason why we imposed essentially the same axioms on the systems of local propositions; we expect that one could determine the Hilbert realization of the CROC-valued propositional systems with an appropriate generalization of the Piron method. Also we consider first this "minimal program" and after the completion of this program we could pass over to the research of the realization of a more general system of axioms in a clear way having the new information provided by the realization of the relatively simpler axioms (A1)–(A10).

The covering law [axiom (A10)] plays an essential role in the Piron realization of the propositional systems. Thus, in accordance with our minimal program, we have imposed on L this axiom, too, although it is not known yet whether this axiom will play a significant role in the determination of the Hilbert realization of the axioms or not. In Section 7, we will see that it is possible to construct such representations with Hilbert modules, in which axioms (A9)–(A10) have no role, i.e., they realize merely the axioms (A1)–(A8).

4. DECOMPOSITIONS OF THE CROC-VALUED PROPOSITIONAL SYSTEMS AND CROC-VALUED LOGICS

One can easily decompose a CROC-valued propositional system into irreducible ones with the use of slightly generalized methods and concepts of Piron's textbook from page 29 to page 35. Here follow only the main concepts and the resulting generalized theorems. It is not possible to give the proofs and all technical details, because this paper would grow much too long. These are left to the reader and to the reference Banai (1980c).

Definition 9. We call a pair of maps (M, m) a c-morphism pair from a CROC-valued logic (L_1, l_1, V_1) into a CROC-valued logic (L_2, l_2, V_2) if M and m are c-morphisms from L_1 into L_2 and from l_1 into l_2, respectively, and for all $v_1 \in V_1$ there exists a $v_2 \in V_2$ such that

$$M \circ v_2|_{M(L_1)} = v_1 \circ m$$

Then the image of a CROC-valued logic under a c-morphism pair is a CROC-valued logic.

> *Theorem 10.* If $(\mathcal{C}, \mathfrak{c})$ is the center pair of a CROC-valued propositional system (L, l, V) then $(\mathcal{C}, \mathfrak{c}, V|_{\mathcal{C}})$ is a classical CROC-valued propositional system.

Definition 11. We shall define the *direct union* of a family $(L_\alpha, l_\alpha, V_\alpha)$ of CROC-valued logics as a CROC-valued logic $(L, l, V) = \bigcup_\alpha (L_\alpha, l_\alpha, V_\alpha)$ obtained in the following manner: L and l are the direct unions of the L_α's and l_α's, respectively: $L = \bigcup_\alpha L_\alpha$, $l = \bigvee_\alpha l_\alpha$, and $V|_{L_\alpha} = V_\alpha$ [where the equality means that $v \in V \Rightarrow \exists \alpha (v|_{L_\alpha} \in V_\alpha)$, and conversely $v_\alpha \in V_\alpha \Rightarrow \exists v \in V(v|_{L_\alpha} = v_\alpha)$].

The direct union of CROC-valued logics is a CROC-valued logic, in fact, and the direct union of CROC-valued propositional systems is a CROC-valued propositional system.

> *Theorem 12.* In $\bigcup_\alpha (L_\alpha, l_\alpha, V_\alpha)$, $\{X_\alpha\} \leftrightarrow \{Y_\alpha\}$, $\{x_\alpha\} \leftrightarrow \{y_\alpha\}$ iff $X_\alpha \leftrightarrow Y_\alpha$, $x_\alpha \leftrightarrow y_\alpha$, respectively. In particular, the center pair of $\bigcup_\alpha (L_\alpha, l_\alpha, V_\alpha)$ is the direct union of the center pairs of the $(L_\alpha, l_\alpha, V_\alpha)$.

Definition 13. A CROC-valued logic (L, l, V) is irreducible if (a) l cannot be split as a direct union of two of its sublattices, each containing more than one element; (b) L cannot be split as a direct union of two of its sublattices, each containing at least one (not necessarily proper) sublattice isomorphic to l with respect to a value function restricted to the sublattice.

> *Theorem 14.* A CROC-valued logic (L, l, V) is irreducible iff its center pair $(\mathcal{C}, \mathfrak{c}) = (\{\theta, \ \}, \{0, 1\})$.

> *Theorem 15.* Every CROC-valued propositional system (L, l, V) is the direct union of irreducible CROC-valued propositional systems.

Theorem 16. A CROC-valued logic (L, l, V) is the direct union of irreducible CROC-valued logics iff its center pair $(\mathcal{C}, \mathfrak{c})$ is atomic (both $\mathcal{C}$ and $\mathfrak{c}$ are atomic).

We can give a weakened irreducibility notion for CROC- (lattice-) valued logics, which may be useful in studying the structure of lattice-valued logics and general orthomodular lattices which are equivalent with lattice-valued logics.

Definition 17. A CROC-valued logic (L, l, V) is *weakly irreducible* if L cannot be split as a direct union of its two sublattices, each containing at least one (not necessarily proper) sublattice isomorphic to l with respect to a value function restricted to the sublattice.

We can prove then the facts about the weak irreducibility with the same methods as those of the above theorems, leaving only the lattice of values l fixed in the notions and concepts (for details and proofs see Banai, 1980c).

Proposition 18. A CROC-valued logic (L, l, V) is weakly irreducible iff, in its center pair $(\mathcal{C}, \mathfrak{c})$, $\mathcal{C} \cong \mathfrak{c}$.

For the proof of this proposition the following lemma is necessary.

Lemma 19. Let $(\mathcal{C}, \mathfrak{c}, V)$ be a Boolean CROC-valued logic. Then

$$(\mathcal{C}, \mathfrak{c}, V) = \bigcup_{\alpha} (\mathcal{C}_\alpha, \mathfrak{c}, V_\alpha)$$

where $\mathcal{C}_\alpha \cong \mathfrak{c}$ and $V_\alpha = \{\text{id}: \mathcal{C} \to \mathfrak{c}\}$ (id means the identity mapping).

Proof. It is sufficient to show that the case $\mathfrak{c} \subset \mathcal{C} \subset \mathfrak{c} \cup \mathfrak{c}$ cannot occur, where also $\mathcal{C}$ is greater than $\mathfrak{c}$ but it is smaller than the direct union of two $\mathfrak{c}$ (defining the inclusion with a sublattice). Such a $\mathcal{C}$ cannot take place in a CROC-valued logic $(\mathcal{C}, \mathfrak{c}, V)$, for, on the contrary, there exists a $Z \in \mathcal{C}$ such that $[\theta, Z] \cong \mathfrak{c}$ and $([\theta, Z], \mathfrak{c}, \{\text{id}\})$ is a CROC-valued logic. Then either $[\theta, \bar{Z}]$ is isomorphic to $\mathfrak{c}$ or it contains a sublattice isomorphic to $\mathfrak{c}$, since $([\theta, \bar{Z}], \mathfrak{c}, V_{[\theta, \bar{Z}]})$ is a CROC-valued logic, too. This contradicts our assumption. The statement follows with (finite or transfinite) induction. ∎

Proposition 20. Every CROC-valued logic (L, l, V) is the direct union of weakly irreducible CROC-valued logics. More precisely

$$(L, l, V) = \bigcup_{\alpha} (L_\alpha, l, V_\alpha)$$

where $\mathcal{C}_\alpha \cong \mathfrak{c}$, $\forall \alpha$.

5. THE CLASSIFICATION OF CROC-VALUED LOGICS AND CROC-VALUED PROPOSITIONAL SYSTEMS

The above theorems enable us to classify the CROC-valued propositional systems and to reduce a large class of these systems to the usual two-valued cases.

(1) $(\mathcal{C}, \mathfrak{c}, V|_{\mathcal{C}})=(L, l, V)$, i.e., (L, l, V) is a classical CROC-valued propositional system, if it satisfy (A9), (A10). In this case

$$(L, l, V)=\bigcup_{\alpha}(L_{\alpha}, l, V_{\alpha})=\bigcup_{\alpha\beta}(L_{\alpha\beta}, l_{\beta}, V_{\alpha\beta}) \tag{5.1}$$

where $(L_{\alpha}, l, V_{\alpha})$'s are weakly irreducible components; L_{α} is isomorphic to l, for all α; further $(L_{\alpha\beta}, l_{\beta}, V_{\alpha\beta})$ is an irreducible component; $L_{\alpha\beta}$ and l_{β} contain only two elements. Also an irreducible component is isomorphic to a classical two-valued irreducible propositional system. If (A9) and (A10) are not satisfied by (L, l, V) then only the first equality holds in (5.1).

(2) $(\mathcal{C}, \mathfrak{c})=(\mathcal{C}, l)$ where $\mathcal{C}$ can be (a) isomorphic to l or (b) bigger than l ($\mathcal{C}\supset l$ by virtue of an isomorphic sublattice of $\mathcal{C}$ to l).

(a) $(\mathcal{C}, \mathfrak{c})=(l, l)$. Then the CROC-valued logic (L, l, V) is weakly irreducible and, if it is at the same time a CROC-valued propositional system (or only l is atomic), we have

$$(L, l, V)=\bigcup_{\alpha}(L_{\alpha}, l_{\alpha}, V_{\alpha})$$

where l_{α} contains two elements and L_{α} is irreducible and one l_{α} corresponds to only one L_{α}. The L_{α}'s, for different α's, are not necessarily isomorphic to each other. Clearly an irreducible component $(L_{\alpha}, l_{\alpha}, V_{\alpha})$ is equivalent to an irreducible propositional system or CROC of Piron (1976). Thus these cases are reduced to the two-valued cases. By analogy we may call such a CROC-valued logic the pure quantum case of a *non*relativistic local field theory (cf. below and Banai, 1980a).

(b) $\mathcal{C}\supset l$. Then

$$(\mathcal{C}, \mathfrak{c}, V|_{\mathcal{C}})=\bigcup_{\alpha}(\mathcal{C}_{\alpha}, l, V_{\alpha})$$

as in (4.1). Thus

$$(L, l, V)=\bigcup_{\alpha}(L_{\alpha}, l, V_{\alpha})$$

where each component of the direct union is a weakly irreducible CROC-valued logic and the center of each L_{α} is isomorphic to l. If $(\mathcal{C}, \mathfrak{c})$ is atomic then

$$(L_\alpha, l, V_\alpha) = \bigcup_\beta (L_{\alpha\beta}, l_\beta, V_{\alpha\beta})$$

we reduce these cases to the usual two-valued ones, too. By analogy we may say in these cases that the nonrelativistic field theoretic system possesses *superselection rules* (cf. below and Banai, 1980a).

(3) $(\mathcal{C}, \mathfrak{c})$ is arbitrary.

Let us consider the cases (a) $\mathcal{C} \cong \mathfrak{c}$ and (b) $\mathcal{C} \supset \mathfrak{c}$.

(a) $\mathcal{C} \cong \mathfrak{c}$. Then the CROC-valued logic (L, l, V) is weakly irreducible and, if $\mathcal{C}$ is atomic,

$$(L, l, V) = \bigcup_\alpha (L_\alpha, l_\alpha, V_\alpha)$$

where the center pair $(\mathcal{C}_\alpha, \mathfrak{c}_\alpha)$ of an irreducible component $(L_\alpha, l_\alpha, V_\alpha)$ is $(\{\theta_\alpha, \mathbb{1}_\alpha\}, \{0_\alpha, 1_\alpha\})$. Such a CROC-valued propositional system in general, if l contains nontrivial elements, cannot reduce to the usual two-valued cases. By analogy again, we may call such a CROC-valued propositional system, or generally CROC-valued logic, the *pure quantum* case of a relativistic local field theory (cf. below).

(b) $\mathcal{C} \supset \mathfrak{c}$. Then

$$(\mathcal{C}, \mathfrak{c}, V|_\mathcal{C}) = \bigcup_\alpha (\mathcal{C}_\alpha, \mathfrak{c}, V|^\alpha_{\mathcal{C}_\alpha})$$

where $\mathcal{C}_\alpha \cong \mathfrak{c}$. Thus

$$(L, l, V) = \bigcup_\alpha (L_\alpha, l, V_\alpha)$$

where each component is a weakly irreducible CROC-valued logic. If $(\mathcal{C}, \mathfrak{c}, V|_\mathcal{C})$ is a classical CROC-valued propositional system, then

$$(L, l, V) = \bigcup_{\alpha\beta} (L_{\alpha\beta}, l_\beta, V_{\alpha\beta})$$

where

$$(\mathcal{C}_{\alpha\beta}, \mathfrak{c}_\beta) = (\{\theta_{\alpha\beta}, {}_{\alpha\beta}\}, \{0_\beta, 1_\beta\})$$

We may say in that case, too, that the relativistic system $P(\Omega)$ has superselection rules (cf. below).

Remark. We see that only those CROC-valued propositional systems and CROC-valued logics having an atomic center pair can reduce to the usual two-valued logic cases (structure-preserving way), in which the set of logical values l is a Boolean lattice. This shows the generality of the CROC-valued logic. (See still Banai, 1980b.)

We see further from Propositions 18 and 20 that if one were able to represent all weakly irreducible CROC-valued logics then the representations of all other CROC-valued logics are the direct union of such CROC-valued logics. This would give a possibility of leaving the axioms (A9)–(A10) which have no direct physical meaning. (The components c_α, isomorphic to the center of l, behave like "atoms" in the center of L.)

6. SYSTEMS OF PROPOSITIONS IN LOCAL FIELD THEORIES

We now show briefly that the two types of noncompatibility relative to measurement processes, mentioned in the Introduction, can be described intrinsically in the CROC-valued propositional systems.

(1) *Classical Physical System* $P(\Omega)$: All propositions are compatible with each other; so L is a distributive lattice. The propositions take their values in $\mathcal{P}(\Omega)$, generally. Thus the propositional system of $P(\Omega)$ is a classical CROC-valued propositional system $(L, \mathcal{P}(\Omega), V)$ which can reduce to the usual two-valued case. [The latter statement is not true if we drop the axioms (A9), (A10) of atomicity.]

(2) *Nonrelativistic Quantum System* $P(\Omega)$, $\Omega \subseteq \mathbb{R}^3$: Two different measurements on the same region S of Ω (or at the same point x of Ω, ideally) can disturb each other, then the corresponding propositions taking the true part of their values on S (or on x) are, in general, *not* compatible. Thus the lattice of propositions L is a nondistributive lattice. On the other hand the measurements in two disjoint subsets S_1, S_2 of Ω (or at two disjoint points) are always compatible, thus the corresponding propositions A_1, A_2 taking the values $v(A_1) = S_1$, $v(A_2) = S_2$ must be compatible, i.e., if $v(A_1) \perp v(A_2)$, $v \in V$ then $A_1 \leftrightarrow A_2$. Also the values of propositions can be represented by $\mathcal{P}(\Omega)$ and the propositional system of $P(\Omega)$ is a CROC-valued propositional system $(L, \mathcal{P}(\Omega), V)$. The requirement of local commutativity follows then from Lemma 5. [If we regard only the local observables whose measured value is a measurable function on Ω then $\mathcal{P}(\Omega)$ reduces to $\mathcal{B}(\Omega)$.] If the center of L is isomorphic to $\mathcal{P}(\Omega)$ then

$$(L, \mathcal{P}(\Omega), V) = \bigcup_{x \in \Omega} (L_x, \{\emptyset, x\}, V_x)$$

where $(L_x, \{\emptyset, x\}, V_x)$ represents a pure quantum system at the point $x \in \Omega$

(in the terminology of Piron, 1976), and this fact suggests the name "pure quantum case of nonrelativistic field theory" for these cases. This makes clear the name "superselection rules in the nonrelativistic case" above (for further details see Banai 1980a).

(3) *Relativistic Quantum System* $P(\Omega)$, $\Omega \subseteq \mathbb{M}^4$: The lattice L is again nondistributive. Einstein causality determines the compatible elements of $l=l(\Omega)$; spacelike separated regions S_1, S_2 (or points x_1, x_2) of Ω must be compatible, then if S_1, $S_2 \in l(\Omega)$, $S_1 \perp S_2$ and $v(A_1)=S_1, v(A_2)=S_2$, $v \in V$ then $A_1 \leftrightarrow A_2$ (cf. Lemma 5). Other regions or points (timelike or lightlike separated ones) need *not* be compatibles. These mean that $l(\Omega)$ has, in general cases, the following lattice structure: if S is a spacelike hyperplane in Ω then it determines a Boolean subalgebra of $l(\Omega)$, and conversely, each Boolean subalgebra of $l(\Omega)$ is isomorphic to a $\mathcal{P}(S)$ where S is a spacelike subset of Ω. It follows further that if $h \in l(\Omega)$ then

$$h' = \{x \in \Omega,\ x \text{ spacelike separated to all } (y \in \Omega \text{ and } y \leqslant h)\}$$

The only elements of $l(\Omega)$ that are compatible with all others are the empty set $\varnothing=0$ and the whole Ω $(=\mathbb{M}^4)$. Thus $l(\Omega)$ is irreducible. We obtained that the CROC-valued propositional system $(L, l, V)=(L, l(\Omega), V)$ is irreducible, if L is irreducible, and may describe a *pure* relativistic quantum system $P(\Omega)$. All other CROC-valued propositional systems (L, l, V) corresponding to relativistic quantum systems $P(\Omega)$ are direct unions of irreducible CROC-valued propositional systems $(L_\alpha, l_\alpha(\Omega), V_\alpha)$, where $l_\alpha(\Omega)$ has the same lattice structure as $l(\Omega)$ above. We may say by analogy that $P(\Omega)$ possesses superselection rules.

7. STATES, SYMMETRIES, AND OBSERVABLES AND THE CONNECTION WITH THE ALGEBRAIC APPROACH

The states, symmetries, and observables can be defined in a CROC-valued propositional system (L, l, V) in a way similar to the corresponding definitions of Piron (1976). The *pure states* of a physical system $P(\Omega)$ can be represented in the corresponding CROC-valued propositional system (L, l, V) [if it exists for $P(\Omega)$] by the atoms of L. The values of the atoms of L in l are the atoms of l. On the other hand, we saw in the preceding section that the atoms of $l(\Omega)$ correspond to the points of the set Ω. Let $x \in \Omega$ be the value of the atom $P \in L$, then P represents the maximal ("pure") information in L obtained from the system at the point x. Conversely, we can represent the maximal information obtained at the point x with an atom P^x of L (the value of P^x is x for a $v \in V$). Thus the principle of locality can be formulated in the CROC-valued propositional system $(L, l(\Omega), V)$ in a simple way. The all pure information obtained from the system $P(\Omega)$ also can be represented by a collection $\{P^x, x \in S \subseteq \Omega\}$ of atoms of L, where the points x are the atoms of a maximal Boolean subalgebra of $l(\Omega)$ [in the nonrelativistic case this Boolean subalgebra is $\mathcal{P}(\Omega) = l(\Omega)$, $S = \Omega$, and the relativistic case it is $\mathcal{P}(S)$, where S is a spacelike subset in Ω]. We may call these collections $\{P^x, x \in S\}$ global pure states. A value function from V belongs to each global pure state and for all propositions (let K_S denote these), compatible with the state $\{P^x, x \in S\}$, we are also able to give the correspondence v: $K_S \to \mathcal{P}(S)$, but we cannot say anything about the values of propositions which are not compatible with the collection $\{P^x, x \in S\}$. Conversely, a value function defines a global pure state for a maximal collection K_S of compatible propositions; the state in which the proposition $A \in K_S$ takes the value $v(A) \in \mathcal{P}(S) \subset l(\Omega)$, the family $\{P^x, x \in S\}$ of the atoms of the collection K_S (K_S is a maximal Boolean algebra of L) gives the global pure state.

A *symmetry* should be a bijective mapping of L onto itself which preserves the LUB and orthocomplementation. Such a bijection is an automorphism, for the inverse map possesses the same properties. A symmetry preserves the compatibility relation and maps the center of L onto itself. At present the exact connection is not known between the symmetries of L and the symmetries of l which can be defined in a similar way. Nevertheless, we expect from the physical point of view that every symmetry of L generates a symmetry of l and conversely, every symmetry of l induces a symmetry of L. It is clear that this means a restriction on the value functions in V. For example, let G be a geometrical symmetry group on the physical space Ω and let $s(g)$ and $S(g)$ be the representations of G among the automorphisms of $l(\Omega)$ and L, respectively. Then the invariance of a value function v under the double action of G on L and on $l(\Omega)$ is expressed by the commutativity of the diagram

$$\begin{array}{ccc} L & \xrightarrow{\;S(g)\;} & L \\ v \downarrow & & \downarrow v \\ l(\Omega) & \xrightarrow{\;s(g)\;} & l(\Omega), \end{array} \qquad v \circ s(g) = S(g) \circ v, \qquad v \in V$$

This condition formally coincides with the imprimitivity condition of Mackey for observables. Now this condition would express the invariance of the state, corresponding to the value function v, under the action of the group G.

A (local) *observable* is defined as a correspondence between the propositions associated with the measuring apparatus $A(\Omega)$ and those associated with the physical system $P(\Omega)$. But the CROC associated with $A(\Omega)$ is not necessarily Boolean and atomic.[6] In the relativistic case, the measurements in two timelike or lightlike separated regions can disturb each other. Thus we should have to associate with $A(\Omega)$ a (non-Boolean) CROC A, where A is a direct union of $a(\Omega)$'s each having a lattice structure similar to $l(\Omega)$ in the relativistic case, the only difference being that they are not necessarily atomic. This is the reason why we shall call a *nonrelativistic* (*local*) *observable* every c-morphism (or more generally, σ-morphism) from a Boolean CROC of subsets of real-valued functions on $\Omega \subseteq \mathbb{R}^3$ into L of the CROC-valued propositional system $(L, \mathcal{P}(\Omega), V)$ and a *relativistic* (*local*) *observable* every c-morphism (σ-morphism) from $A = \bigcup_i a_i(\Omega)$, $\Omega \subseteq \mathbb{M}^4$ into L of the CROC-valued propositional system $(L, l(\Omega), V)$. Then a nonrelativistic observable determines the spectral decomposition of a real-valued function, with respect to a value function v in V, and a relativistic observable determines the spectral decomposition of a family of self-adjoint operators, with respect to a value function v in V. This real-valued function on $\Omega \subseteq \mathbb{R}^3$, respectively, self-adjoint operators on the Hilbert space $H(\Omega)$ associated with $\Omega \subseteq \mathbb{M}^4$ (cf. Section 7) is the measured value and values of the nonrelativistic observable, respectively, relativistic observable. (Each measured value of a relativistic observable, with respect to a value function, is concentrated on a spacelike subset of Ω, on which it can be measured without dispersion.)

We note that the states and the connection of them with the probabilistic description and the symmetries and the nonrelativistic observables are studied in more detail in the nonrelativistic case by Banai (1980a).

Now we can easily see that the postulates of isotony and local commutativity of Haag and Kastler (1964) (or see Emch, 1972) for local field theories are satisfied naturally by our propositional systems; further the space-time covariance can be implemented in a natural way in these propositional systems. From the definitions of observables and symmetries follows that the algebra of local observables of a physical system $P(\Omega)$ is generated by the CROC-valued propositional system (L, l, V) of $P(\Omega)$ and the symmetries of the local algebra are represented by automorphisms of this algebra. Thus it is sufficient to show that the postulates hold for the CROC-valued propositional systems (L, l, V).

Let b be an open set with compact closure in the configuration space ($\mathbb{R}^3$ or $\mathbb{M}^4$) and Ω be the set theoretic union of b's. Then $(L_b, l(b), V_b)$ denotes the CROC-valued propositional system of local propositions in the

[6]We do not have to associate with $A(\Omega)$ a CROC-valued logic, as we can easily see (cf. the classical case in Section 2).

region b. Now $(L_b, l(b), V_b)$ generates the algebra $A(b)$ of local observables in b, and the CROC-valued propositional system $(L, l(\Omega), V)$ generates the set theoretic union of $A(b)$'s (and its completion), A. It follows from the construction of the CROC-valued logic $(L, l(\Omega), V)$ that (see Banai, 1980c)

$$(L_b, l(b), V_b) = ([\theta, B^b], [0, b], V_{[\theta, B^b]})$$

where $[\theta, B^b]$ and $[0, b]$ are in order the segments of L and $l(\Omega)$ with the relative orthocomplementations. $(L_b, l(b), V_b)$ is a sub-CROC-valued logic of $(L, l(\Omega), V)$ (see Banai, 1980c); the corresponding $A(b)$ is a subalgebra of A. Now we have the following.

(1) Isotony: if $b_1 \subset b_2$ then

$$(L_{b_1}, l(b_1), V_{b_1}) \subset (L_{b_2}, l(b_2), V_{b_2})$$

what satisfies in $(L, l(\Omega), V)$, in fact.

(2) Local commutativity: if $b_1 \perp b_2$ then

$$(L_{b_1}, l(b_1), V_{b_1}) \leftrightarrow (L_{b_2}, l(b_2), V_{b_2})$$

i.e., the elements of L_{b_1} are compatible with the elements of L_{b_2} in L as we saw above (Lemma 5.).

(3) Covariance: Let G denote either the Euclidean group $\mathbb{E}^3$ or the inhomogeneous proper Lorentz group $L_+^\uparrow$, according to the nonrelativistic, or respectively, relativistic case. Then G is represented by automorphisms $A \in L \to A^g \in L$, $g \in G$ from the definition of symmetries. The covariance postulates now are equivalent to the requirement

$$(L, l(b), V_b)^g = (L_{gb}, l(gb), V_{gb})$$

in $(L, l(\Omega), V_G)$, where gb is the image of the region b under the action of g and the elements of V_G are compatible with the symmetry group G (they are invariant under the action of G).

It is clear that one could control the other postulates of Haag and Kastler (1964, postulates 1, 4, and 6), in which they postulate the C^*-algebraic properties and structure of $A(b)$'s and A, with the determination of the algebraic structure of local algebras generated by $(L_b, l(b), V_b)$'s and $(L, l(\Omega), V)$ (see Banai, 1980a).

A further note is that the axioms (A9) and (A10) have no substantial role in the above considerations; one could use only CROC-valued logics instead of CROC-valued propositional systems in a more general consideration [in that case we can define, for example, the states of the physical system with the maximal filters of L (instead of the atoms of L)].

8. REALIZATION OF CROC-VALUED LOGICS

Finally let us give some examples of realizations of CROC-valued logics with topological, especially with Hilbert modules, which are among the most common structures for realizations.

(1) (L, l, V) *is Classical*. If (L, l, V) is a CROC-valued propositional system, then an irreducible component of it is isomorphic to a set with two elements and (L, l, V) is a direct union of such sets. On the other hand, there exists a set Γ to l such that $l = \mathcal{P}(\Gamma)$ and Γ is the set of the atoms of l (Piron, 1976). Then we can easily see that $(\mathcal{P}[\mathcal{P}(\Gamma)], \mathcal{P}(\Gamma), V)$ is a CROC-valued propositional system, where the elements of V are generated by the real-valued functions $\Gamma \to \mathbb{R}$ (if Γ is countable). It is clear that $\mathcal{P}[\mathcal{P}(\Gamma)]$ is a propositional system. Now, as is well known from the set theory, $\mathcal{P}(\Gamma)$ can be represented by the set $R(\Gamma)$ of all real-valued functions on Γ. Then $\mathcal{P}(\mathcal{P}(\Gamma)) = \mathcal{P}(R(\Gamma))$. Let $S(\Omega)$ be representable with a mapping $\Gamma \to \mathcal{P}(\mathbb{R})$; $x \to i_x$. Now if $f \in R(\Gamma)$ then define the mapping v_f: $\mathcal{P}(R(\Gamma)) \to \mathcal{P}(\Gamma)$ the following way:

$$v_f[S(\Omega)] = \omega := \{x \mid x \in \Gamma, f(x) \in i_x = S(\Omega)\}$$

One can easily verify that v_f satisfies the definition of a value function (Definition 3.). Conversely, if v is a value function from $\mathcal{P}(R(\Gamma))$ to $\mathcal{P}(\Gamma)$ and Γ is countable then v generates a function $f \in R(\Gamma)$ because v is a unitary c-morphism (see Piron, 1976).

Now let $\omega \in \mathcal{P}(\Gamma)$ be arbitrary and let $R(\omega)$ be the set of those elements of $R(\Gamma)$ which map the $C\omega$ to 0 ($C\omega$ is the complement of ω). Then $R(\omega) \in \mathcal{P}(R(\Gamma))$ and $v(R(\omega)) = \omega$, $\forall v \in V$. Also $(\mathcal{P}(\mathcal{P}(\Gamma)), \mathcal{P}(\Gamma), V)$ is a CROC-valued propositional system, in fact.

Now let (L, l, V) be a classical CROC-valued logic and Γ be a set; further let $b(\Gamma)$ be a CROC from the subsets of Γ such that $l = b(\Gamma)$. Let us consider $R(\Gamma)$; then L can be embedded into a CROC of subsets of $R(\Gamma)$. Let $B(R(\Gamma))$ be a CROC such that the elements of $B(R(\Gamma))$ take their values in $b(\Gamma)$ and if $\gamma \in b(\Gamma)$ then $R(\gamma)$ (the set of functions mapping $C\gamma$ into 0) is an element of $B(R(\Gamma))$. Then $L = B(R(\Gamma)) = \mathcal{P}(R_b(\Gamma))$, where $R_b(\Gamma)$ is a subalgebra of $R(\Gamma)$ generated by $b(\Gamma)$. The elements of V are generated by the elements of $R_b(\Gamma)$ (if Γ is countable); thus $(L, l, V) = (\mathcal{P}(R_b(\Gamma)), b(\Gamma), V_b)$.

Let us consider algebraically the structures under consideration. $R_b(\Gamma)$ generated by $b(\Gamma)$ is a subring of the ring $R(\Gamma)$. On the other hand, $B(R(\Gamma))$ generates a subalgebra $M_B[R(\Gamma)]=M[R_b(\Gamma)]$ of the set $M[R(\Gamma)]$ of functions from $R(\Gamma)$ to $R(\Gamma)$. $M[R(\Gamma)]$ is a module over $R(\Gamma)$ and $M_B[R(\Gamma)]$ is a submodule of $M[R(\Gamma)]$ and $M[R_b(\Gamma)]$ is a module over $R_b(\Gamma)$. Also, in general, the classical (L, l, V) can be embedded into the pair $(M[R(\Gamma)], R(\Gamma))$, where $M[R(\Gamma)]$ is a module over $R(\Gamma)$.

(2) (L, l, V) *Describes a Pure Nonrelativistic Quantum Case* $((\mathcal{C}, c)=(l, l))$. If l is atomic then (L, l, V) can be realized by a direct integral of Hilbert spaces (Piron, 1976). Another example is provided by the AW^*-module of Kaplansky (1953). Let H_A be a faithful AW^*-module (which is also a Hilbert module in our terminology) over A (A is a commutative AW^*-algebra). Then the set of AW^*-submodules of H_A (denoted by $\mathcal{P}[H_A]$) is a CROC. By definition, the ordering relation is the set theoretic inclusion relation; every intersection of AW^*-submodules is an AW^*-submodule, which implies the existence of a GLB. The mapping which brings an AW^*-submodule into correspondence with its orthogonal complement, is an orthocomplementation. The weak modularity relation can be verified immediately by passing over to orthogonal projectors of the AW^*-submodules. Let A be a commutative AW^*-algebra such that the lattice of its self-adjoint projectors $l(A)$ is a distributive CROC, and H_A be an AW^*-module over A. Let now $l(A)$ be atomic and p denotes an atom in it. Further let H_A be a homogeneous AW^*-module with orthonormal basis $\{x_\lambda\}$ and P be a one-dimensional submodule of H_A ($\{ax_\gamma\}, a\in A$ and $x_\gamma\in\{x_\lambda\}$). Then pP represents an atom in $\mathcal{P}(H_A)$. Now (A9) satisfies trivially in $\mathcal{P}(H_A)$ and (A10) can be checked with a similar argument as in the case of Hilbert spaces. Let $B=\mathcal{P}(\Omega)=l(A)$ [in the function algebra $C(\Gamma)$, representing A, now Γ is a Stonean space belonging to B and B is atomic, which means that the set of isolated points of Γ is dense in Γ; $\bar{\Omega}=\Gamma$). Thus we get that in $(\mathcal{P}(H_A), \mathcal{P}(\Omega), V)$ $\mathcal{P}(H_A)$ and $\mathcal{P}(\Omega)$ are propositional systems, and an element of V generates a function in $R(\Gamma)$, restricted v to a maximal Boolean subalgebra of $\mathcal{P}(H_A)$ (and if Ω is countable). Conversely, every $f\in R(\Gamma)$ defines a value function, mapping into $\mathcal{P}(\Omega)$, on a maximal Boolean sublattice of $\mathcal{P}(H_A)$ in a way described under (1) above. [This mapping certainly coincides with a value function from $\mathcal{P}(H_A)$ onto $\mathcal{P}(\Omega)$, but not necessarily in a unique way.]

Now let ω be an element of $\mathcal{P}(\Omega)$ and e_ω be the projector in A corresponding to ω; then $e_\omega H_A$ is an element of $\mathcal{P}(H_A)$ and $v(e_\omega H_A)=\omega$, $\forall v\in V$. Since H_A is faithful with respect to A thus for all $\omega\in\mathcal{P}(\Omega)$ $e_\omega H_A\not\equiv 0$ ($\omega\neq 0$). We obtained that $(\mathcal{P}(H_A), \mathcal{P}(\Omega), V)$ is a CROC-valued propositional system and weakly irreducible since $\mathcal{P}(\Omega)$ is isomorphic to the center of $\mathcal{P}(H_A)$ (from Theorem 7 of Kaplansky, 1953).

If we assume not that B is atomic and H_A is homogeneous, then $(\mathcal{P}(H_A), B, V)$ is only a weakly irreducible CROC-valued logic (see Banai, 1980c).

This example suggests that we use H_A, in the description of a nonrelativistic (local) quantum system $P(\Omega)$, in a similar way as a Hilbert space is used in the Hilbert space formulation of a quantum mechanical system. (See in detail Banai, 1980a).

(3) By generalizing the above example, we should like to determine the Hilbert realization (i.e., with Hilbert modules over C^*-algebras) of the all irreducible CROC-valued propositional systems and weakly irreducible CROC-valued logics, but in the present stage of our work, we have only conjecture for the representations of the CROC-valued propositional systems in the pure relativistic quantum cases. [See still for general conjectures and expectations of the Hilbert realization of CROC-valued logics Banai (1980c).]

Briefly this is the following: $l(\Omega)$ is irreducible, thus we can represent it with closed subspaces of a Hilbert space $H(\Omega)$. This means that we should impose on Ω ($\stackrel{?}{=} \mathbb{M}^4$) a Hilbert space structure with a bilinear inner product $\langle , \rangle$. Then the Boolean subalgebras in $\mathcal{P}[H(\Omega)] = l(\Omega)$ are determined by the spacelike subsets of Ω. The orthocomplement of an $x \in \Omega$ is the set of all $x' \in \Omega$ with $g_{\mu\nu}(x-x')^\mu(x-x')^\nu < 0$. This gives that the inner product should have the property $\langle x, x' \rangle = 0$ for all $x, x' \in \Omega$, $g_{\mu\nu}(x-x')^\mu(x-x')^\nu < 0$.

Let $A(\Omega)$ be a C^*-algebra generated by the orthogonal projectors of $H(\Omega)$ [then $A(\Omega)$ is a factor and, when $l(\Omega)$ is atomic, of type I]. Then any commutative subalgebra of $A(\Omega)$ is isomorphic to a $C(S)$, where S is a spacelike subset of Ω.

Let $H_{A(\Omega)}$ be a Hilbert module over $A(\Omega)$, i.e., an inner product taking values in $A(\Omega)$ exists on $H_{A(\Omega)}$ and this inner product generates an A-norm on $H_{A(\Omega)}$ and $H_{A(\Omega)}$ is complete with respect to this A-norm [see for exact definitions Banai (1978, 1980a)]. Then the set $\mathcal{P}[H_{A(\Omega)}]$ of all closed submodules of $H_{A(\Omega)}$ is an irreducible CROC, and $(\mathcal{P}(H_{A(\Omega)}), \mathcal{P}(H(\Omega)), V)$ is an irreducible CROC-valued logic if $H_{A(\Omega)}$ is faithful. The eigenvalues of the orthogonal projectors of the elements of $\mathcal{P}[H_{A(\Omega)}]$ are the self-adjoint projectors of $A(\Omega)$.

We see that the first problem along this line of thought is the determination of the exact connection between the lattice of projectors of $A(\Omega)$ (or, equivalently, the lattice $l(\Omega)$ of values of the local propositions) and the structure of space-time. We should like to discuss this problem in more detail in another paper.

Remark. Here we would like to mention an interesting related work of G. Takeuti (1979). Takeuti introduced the quantum set theory as a set theory based on quantum logic. He generalized the Boolean-valued model of set theory to an L-valued model, where L is the lattice of all closed linear subspaces of a Hilbert space. It is clear that there is a close connection between the quantum set theory of Takeuti and the above-developed lattice-valued logics; for $l=L$, the CROC-valued logics (L, l, V) give special (and differently defined) examples for L-valued models and the topological modules (thus the Hilbert modules, the most favorite candidate for realizing CROC-valued logics) give the characteristic structures of the universe $V^{(L)}$ of quantum set theory. Thus we can develop the theory of functional analysis of topological modules based on the mathematics that follows from quantum set theory.

We note also that we developed here the theory of lattice-valued logics, independently of the work of Takeuti, from a physical point of view considering the measuring processes of a local field theoretic system. It follows also from this viewpoint that the mathematics used by the well-defined relativistic quantum theory of local fields will have to be in deep connection with the mathematics based on quantum set theory.

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